Behind the Building Blocks

Commodities and Individuals in General Equilibrium Theory

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INTRODUCTION

Every theoretical model of an economic system uses basic building blocks. They are primitive concepts used to develop other parts of the model. They must be completely consistent with the concepts used by the model, and they must be compatible with the theoretical objective of the model.

In the case of general equilibrium theory, the construction of the classic Arrow-Debreu model (Arrow 1951, Debreu 1951, Arrow and Debreu 1954) starts with the concepts of commodities, prices, and then proceeds to define the economic agents involved in the economic model, consumers and producers.

The primitive concepts of commodities and prices, consumers and producers, have not attracted much attention. Most economists consider the analysis of these primitive concepts part of arcane discussions about value theory and, as such, they are seen as involving metaphysical problems or questions solved a long time ago during the infancy of the discipline. This explains why, fifty years after the appearance of the Arrow-Debreu model, little or no attention has been accorded to the detailed analysis of the economic significance of these concepts. On the other hand, the Arrow-Debreu model did inspire many economists to attempt to relax its more restrictive or cumbersome assumptions, some of which are imposed by its use of certain mathematical tools.[2]

Why is the study of the concept of commodities and prices important today? Because these concepts are the building blocks of the most important theoretical model of the market economy that we have available today. But we have to answer here the question: what is at stake? What is the relevance of this question today?

This paper centers on the validity of the procedures followed to build these primitive concepts in the Arrow-Debreu general equilibrium model, a model which uses “the most important mathematical devices in mathematical economics” (Geanakoplos 1989). The first section focuses on the concepts of commodities and prices while a second section concentrates on the definition of consumers and producers. Our concluding remarks consider the implications of this analysis for the proof of existence of a general competitive equilibrium.

COMMODITIES AND PRICES

A commodity is a primitive concept (Debreu 1991) and the model comprises a finite set of classes of commodities. The number of distinguishable commodities (Debreu 1959) or commodity labels (Koopmans and Bausch 1959) is a natural number. The Arrow-Debreu model continues by defining these commodities as goods that are physically determined (Debreu 1982). This tradition goes back as far as the origins of economics as an autonomous discipline, and in the context of general equilibrium theory, it is firmly endorsed since Walras (1952).

But after asserting that commodities are physically determined, in a number of striking passages Debreu (1959: 30) states that their quantities can be expressed as real numbers. This of course poses a problem because it means that irrational numbers can be used to express quantities of physical objects, even indivisible things.

Debreu (1959: 28-29) states that "[a] commodity is characterized by its physical properties, the date at which it will be made available, and the location at which it will be made available". He then asserts that "[w]ith each commodity is associated a real number, its price".

As for services, they are also "goods" (Arrow and Hahn 1971) and Debreu also considers them in terms of physical characteristics, as well as location and date of availability. As in the case of material commodities, for services Debreu states that "[t]heir quantities can, by assumption, be any real numbers" (ibid.: 32).

Even the use of rational numbers to denote quantities of physical objects implies assuming that these physical objects are perfectly divisible. This, in turn, means that no matter how far we divide the physical objects, we still obtain objects with the same physical properties. Although this may appeal to our intuition when it comes to milk or flour, it is completely devoid of sense when it comes to physical objects that come in discrete units. Who among us owns exactly 1.379 cars, or gets 2.408 haircuts? The use of real numbers, including irrational numbers, implies that consumers can somehow specify quantities of goods that are not even fractions. This diverges even farther from experience and common sense. After all, nobody goes to the local Wal-Mart to purchase $\sqrt{2}$ vacuum cleaners, or π PC's.

As he discusses some examples, it is easy to observe that Debreu himself is aware of this conundrum (ibid.: 30):

"A quantity of well-defined trucks is an integer; but it will be assumed instead this quantity can be any real number. This assumption of perfect divisibility is imposed by the present stage of development of economics; it is quite acceptable for an economic agent producing or consuming a large number of trucks."

Debreu concludes: “a commodity is a good or service completely specified physically, temporally, and spatially. (...) It is also assumed that the quantity of any one of them can be any real number.”

This conclusion notwithstanding, Debreu hesitates and recants in footnote (3) to the chapter on prices and commodities: “Two important and difficult questions are not answered by the approach taken here: the integration of money in the theory of value and the inclusion of *indivisible commodities*”. (Our emphasis. See Benetti, in this volume, on the integration of money into general equilibrium theory).

This footnote is important in the light of a comment by Debreu on the axiomatic form of the logical discourse followed in economics: “When [economics] acquires an axiomatic form, its explicit assumptions delimit its domain of applicability and make illegitimate overstepping of its boundary flagrant”. (Debreu 1991) As we shall see, the need to use real numbers to model quantities of physical objects is neither the result of an economically meaningful assumption, nor is it imposed by the present stage of development of economics. It is imposed by mathematics.

Debreu (1991) admits that mathematics is a very demanding master. And that can appear as an understatement when one realizes that the assumptions of his model, made on the grounds of mathematical convenience, limits the scope of the proof of existence of a general competitive equilibrium to barter economies with perfectly divisible commodities, a curious case that may be devoid of theoretical relevance.

At such a high cost, why would a model of a market economy use real numbers to denote the quantities of physical commodities? The best explanation is provided by Debreu himself. According to Debreu (1991: 3) “the central concept of the quantity of a commodity has a natural linear structure”. Thus, actions of agents can be described by listing the quantity of its input or output for each commodity, and that list can be treated as the list of coordinates of a point in the linear commodity space. The price system can be conceived as a point in the linear price space, dual of the commodity space. According to Debreu (ibid.) “In those two linear spaces, the stage was set for sometimes dazzling mathematical developments that began with the elements of differential calculus and linear algebra and that gradually called on an ever broader array of powerful techniques and fundamental results offered by mathematics.”

Among the roles of prices that were illuminated by basic mathematics, Debreu quotes “the achievement of an efficient use of resources, by results of convex analysis” and “the equalization of supply and demand for commodities, by results of fixed point theory”. We know of course that separation theorems, as well as fixed-point theorems, are valid in the space of real numbers, but not in a space restricted to rational numbers (let alone integers). So, the dazzling mathematical developments that called for more powerful techniques appear to be the source of our need to appeal to real numbers for the commodity space and its dual, the price space.

To each commodity is associated its price, which is a real number. The price space is conceived as the dual of the commodity space (Debreu 1991). Because the key conclusions of the model rest on theorems that are valid only in the space of real numbers, the value of economic actions must be able to take on any real value within the relevant range. Thus it is impossible to confine prices, quantities, or the products of prices and quantities, to rational numbers, let alone integers. Unfortunately, we are stuck with the need to justify the use of irrational numbers such as $\sqrt{2}$, π or e, as economically meaningful prices and/or quantities.

One of the theorems that require the use of real numbers is the maximum theorem (Weierstrass' theorem), which is needed to guarantee the mathematical definition of individual supply and demand functions. The other is the fixed-point theorem for upper semi-continuous correspondences (Kakutani's theorem).

The first of these theorems is employed in the construction of individual agents (consumers and producers), which are endowed with maximization functions. Weierstrass' theorem states that given *S* compact and non-empty, and a function *f* from $S \to$ R, with *f* continuous on *S*, and then *f(S)* has a maximum and a minimum. (A closed, bounded interval is compact in the space of real numbers, but not rational numbers; thus the need for irrational numbers of vacuum cleaners and everything else.)

For each producer, there is a well-defined vector **y** of activity (inputs and outputs denoted with a negative and positive sign, respectively); the product **p•y** is the value of the producer's profits. Vector **y** is selected to maximize p y over the production possibility set Y_k for producer k, where Y_k is a compact, convex, non-empty set of points in the R_l commodity space. This ensures that **p•y** attains its maximum value over Y_k for any vector of prices. The supply function and the production possibility set have the required properties so that Weierstrass' theorem can be applied.

In the case of consumers, maximization takes place with respect to a preference ordering $\succsim$ on a subset X of R_l and the maximization problem consists in the selection of a well-defined consumption menu $\mathbf{x}_i$ that is at least as preferred as all the other consumption menus in the consumption possibility set X_i that respect the budget constraint. The existence of a most preferred element is guaranteed if the consumer's set of possible consumption menus X_i in R_l is compact, and if the preference ordering $\succsim_i$ is continuous. (Nikaido 1968)

Thus, in the case of both consumers and producers, commodity bundles are subject to a measure operation determined by the behavioral rule inherent to these economic agents.

This measure operation involves magnitudes that need to be expressed in terms of real numbers.

To conclude this section, we need to clarify how these quantities of heterogeneous goods are measured in value terms. Debreu (1959: 32) states that “[w]ith each commodity, say the *h*th one, is associated a real number, its price p_h. This price can be interpreted as the amount paid now by an agent for every unit of the *h*th commodity which will be made available to him”. But Debreu never clarifies what is it that the agent will pay in exchange for a unit of the *h*th commodity. Based on the standard conventions of economic theory, it would appear that prices are expressed in a common unit of account, or a *numéraire*.

We know of course that the introduction of money in the general equilibrium model poses several problems (see Benetti in this volume) and Debreu himself clarifies in a footnote that his model does not tackle the “difficult issue of money”. So if money is not what agents use in transactions, and if commodities are physically-determined goods, we need to know more about how agents calculate the value of their economic actions.

Debreu does not dwell much on these issues and simply states that “[t]he price system is the *l*-tuple $\mathbf{p} = (p_1,, p_2, ... p_l)$; it can clearly be represented by a point of R_l. The value of an action a relative to the price system p is $\sum p_h a_h$, i.e., the inner product $\mathbf{p} \bullet \mathbf{a}$.”

This leaves two options. First, let commodities be physical entities, and their quantities be expressed by real numbers to which specific dimensions are associated. In that case, the price system cannot be expressed in terms of dimension-less pure numbers. If **p** is a vector whose elements are the dimension-less numbers denoting the prices of the *l* commodities, calculating the value of an economic action through the operation **p•a** cannot be carried out because the sum $p_1x_1 + p_2x_2 + \ldots + p_nx_n$ cannot be performed.

The Arrow-Debreu model uses a normalization procedure for the price system. But this does not solve the problem. The normalization procedure is related to the property of homogeneity of degree zero for the supply and demand functions. And the normalization procedure ensures that each price system p has the following property: $\sum p_i = 1$.

If relative prices are pure or dimension-less numbers, then $\sum p_i = 1$ makes mathematical sense. But in that case, given the fact that commodities are expressed in terms of the units specific to the physical characteristics of goods, it is impossible for agents to calculate the value of a given economic action **p•a** because that inner product is not defined.

The second option relies on the expression of prices in terms of a unit of account that renders these dimensions homogeneous. Prices must now be expressed as a normalized set of ratios defined as physical rates of substitution between goods and are marked by a composite dimension: $p_{i,h}$ is the price of commodity i in terms of commodity h, where h is the unit of account.

In this case, the normalization condition $\sum p_i = 1$ needs to be expressed in terms of relative prices. But here we encounter a difficulty. This normalization condition cannot be carried out in the case of prices taken as physical rates of substitution. Although the normalization condition does not need to have a precise or definite economic interpretation, it must be intelligible from the mathematical point of view. In other words, it must be a well-defined mathematical proposition. As we shall see, this is not the case.

Because the equilibrium price of the commodity chosen as *numéraire* needs to be positive in equilibrium, a composite commodity that will always have this property is chosen as *numéraire*. A composite commodity N made of a unit of each and every good in the economy is sufficient to have this property.

The price vector is expressed in terms of N as follows:

$$p = [\alpha_1 \cdot N / u_1, \alpha_2 \cdot N / u_2, ..., \alpha_l \cdot N / u_l]$$

where each component corresponds to the fraction α of the composite commodity N that is exchanged for one unit of each commodity i.

By definition, $p \cdot N = 1N$ (denoting the price of the *numéraire* in terms of itself) and this implies that $\sum \alpha_i = 1$.

The normalization condition that is imposed by the use of mathematical theorems which are defined for compact sets requires that all price vectors remain in the unit simplex of R_l^+. These price vectors have the property $\sum p_i = 1$ and they can now be explicitly written as follows:

$$\sum p_i = [\alpha_1 \cdot N / u_1 + \alpha_2 \cdot N / u_2 + \cdots + \alpha_l \cdot N / u_l] = 1$$

The problem now is that this sum is not intelligible because for each element in that sum we have a composite dimension (for every commodity i, α_i units of the composite commodity N per unit of i). This composite dimension expresses a physical rate of substitution. The composite dimension does not disappear in the normalization procedure and the addition of heterogeneous elements is impossible. And although the normalization procedure is an abstract operation that may be devoid of economic meaning, it must make mathematical sense.

Summarizing, when agents need to calculate the value of an economic action, say vector **x**, then they are confronted with the following possibilities:

The quantities of physical commodities and the elements of the price vector are expressed in terms of pure dimension-less numbers, and the interior product **p•a** is mathematically defined, but the economic sense is unclear; or the quantities of physical commodities and their (relative) prices are both expressed in terms of their dimensions. This makes sense in so far as the numbers associated with specific dimensions cancel out and in the end we have the total value of an economic action expressed in terms of the unit of account. But the normalization operation $\sum p_i = 1$ cannot be carried out.

Therefore, in order to dispense with the second difficulty, general equilibrium theory makes the following choice: the quantities of commodities, as well as prices, are expressed in pure dimensionless numbers. The interior product **p•a** is then well defined. But dimensionless commodities and prices are entities devoid of physical units: it is not possible to claim that commodities are "goods or services completely specified physically, temporally, and spatially".

INDIVIDUAL AGENTS

The economy that is described by general equilibrium theory is made up of multiple individual agents. There are two classes of agents: producers and consumers. In the

Arrow-Debreu model each individual agent “is characterized by the limitations on his choice, and by his choice criterion” (Debreu 1959: 37). The theoretical problem involved here concerns the possibility of constructing, in a logically consistent manner, the individual agents that interact in the model of a decentralized private economy. They must behave like private agents (they only possess information about their technologies and preferences) and in the pursuit of their self-aggrandizing goals, they are not coordinated by a central authority. Of course, the usual assumption about price takers holds.

The problem with the conception of individual agents, as defined by the theory, is that there may be no natural bound for their activities. If a producer makes positive profits and operates under constant or non-decreasing returns to scale, the profit maximizing strategy is to expand without limit unless the individual possibility sets are bounded. At the level of the overall economy, the set of feasible allocations is bounded because there are no produced resources used as inputs that are unlimited in quantity. But how can individual agents know what are the bounds of the economy? If individual sets are bounded in an arbitrary manner, the equilibrium allocations may be unattainable. For consumers, the argument is similar: if the individual possibility set is not bounded, there may be no preferred element for a given price vector. In addition, the budget constraint of consumers may be undetermined because it incorporates their share of firms’ profits, which may not be defined. Let’s examine these problems in detail.

Producers are economic agents whose role is to choose a production plan. There are m producers and each one of them is given an index k = 1, ..., m). A production plan specifies the quantities of inputs (with negative numbers) and outputs (positive numbers). A production plan is a point in R^n, the commodity space. The set Y_k of all the possible production vectors $\mathbf{y}_k$ for producer k is a sub-set of R^n. In addition, $0 \in Y_k$. This means that every producer k has the possibility of choosing the vector of zero activity.

There are several important properties of Y_k. Y_k is a closed set in R^n meaning that it contains all its own limit points. Also, Y_k is convex in R^n: if $\mathbf{y}^1 \in Y_k$ and $\mathbf{y}^2 \in Y_k$, then $t\mathbf{y}^1 + (t - 1)\mathbf{y}^2 \in Y_k$, for $0 \leq t \leq 1$. This means that if two production vectors are possible for producer k, so is their weighted average with arbitrary positive weights. Particularly important is the following property for individual production possibility sets: Y_k can be a cone with vertex 0. This means that if $\mathbf{y} \in Y_k$, then $\alpha \mathbf{y}_k \in Y_k$, $\alpha > 0$.

These properties are conserved in aggregation and are thus important for the aggregate production possibility set of the entire economy, $\sum Y_k = Y$: Y is closed and convex, and $0 \in Y$.[3] Also, free production is impossible, $Y \cap \Omega = \{0\}$; and in the aggregate, production plans are irreversible, $Y \cap (-Y) = \{0\}$.

Finally, producers are endowed with a behavioral rule that enables them to maximize profits. This rule allows them to choose a production plan that maximizes profits and is given by the supply and profit functions $\varphi_k(\mathbf{p})$ and $\pi_k(\mathbf{p})$ for each producer k:

$$\varphi_k(\mathbf{p}) = \{y_k \mid \mathbf{p}\cdot\mathbf{y}_k = \max \mathbf{p}\cdot\mathbf{y}_k \text{ over all } \mathbf{y} \in Y_k\}$$

$$\pi_k(\mathbf{p}) = \max \mathbf{p}\cdot\mathbf{y}_k \text{ over all } \mathbf{y} \in Y_k \quad (k = 1, \ldots, m).$$

All producers consider prices **p** as given and choose production vectors $\mathbf{y}_k$ so that the product $\mathbf{p}\mathbf{y}_k$ provides the maximum profit.

Consumers choose consumption plans $\mathbf{x}_i$ from their possibility sets X_i. The elements of consumption plans $\mathbf{x}_i$ have a positive sign if they are inputs for the consumer and a negative sign if they are supplied by the consumer. As in the case of producers, the set X_i has certain important properties. First, X_i is a convex set in R^n: if $\mathbf{x}_1, \mathbf{x}_2 \in X_i$ then $t\mathbf{x}_1 + (1 - t)\mathbf{x}_2 \in Xi$ for $0 \leq t \leq 1$. If two consumption sets are possible for consumer i, then the weighted average or linear combination of these two consumption vectors is also possible for consumer i. Second, X_i is closed in R^n. Finally, each X_i has a lower bound $\mathbf{c}_i$ which satisfies $\mathbf{x}_i \geq \mathbf{c}_i$ for all $\mathbf{x}_i \in X_i$.

The elements of each consumption possibility set X_i are ordered through a preference relation $\succsim_i$. The preference ordering is complete, i.e., given any two consumption vectors or commodity bundles $\mathbf{x}_1$ and $\mathbf{x}_2$, the consumer i will be able to order them under $\succsim_i$. The pair $(X_i, \succsim_i)$ is a preference field.

In addition, the preference ordering $\succsim_i$ is convex, i.e., given $\mathbf{x}_1 \succsim_i \mathbf{x}_2$ for $\mathbf{x}_1$ and $\mathbf{x}_2 \in$ Xi, then $t\mathbf{x}_1 + (1 - t)\mathbf{x}_2 \succsim_i \mathbf{x}_2$. Finally, the preference ordering is closed.

The aggregate consumption possibility set for the entire economy is $\sum X_i = X$. Some of the properties of X_i are conserved in aggregation. Thus, X is closed and convex, and has a lower bound for $\leq$ .. Each consumer i has an initial endowment of goods $\mathbf{a}_i$ which is a vector of R^n. Finally, there are *lm* constants $\alpha_{ik} \geq 0$ which represent the share of the i-th consumer to the profits of the k production firms. It is assumed that all profits are distributed in this manner so that $\sum_i \alpha_{ik} = 1$ (k = 1, ... , m).

Consumers are specified as preference fields $(X_i, \succsim_i)$ and their behavior is defined by the demand function $\phi_i(\mathbf{p})$: given a price vector $\mathbf{p}$, consumer i selects the most preferred commodity bundle among those that satisfy the budget constraint:

$$\phi_i(\mathbf{p}) = \{\mathbf{x}_i \mid \mathbf{x}_i \in X_i, \mathbf{x}_i \succsim_i \mathbf{x} \text{ for all } \mathbf{x} \in X_i \text{ subject to } \mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}\cdot\mathbf{a}_i + \textstyle\sum_{k=1} \alpha_{ik} \pi_k(\mathbf{p})\} \ .$$

Once these individual agents have been specified, the model is ready to describe the workings of the market mechanism.

However, in spite of all the assumptions that have been introduced, there are new difficulties arising from the fact that the properties of the individual production possibility sets Y_k are not sufficient to guarantee the existence of production vectors $\mathbf{y}_k$ that maximize profits. Thus, the profit function is not defined and it is not possible to guarantee $\pi_k(\mathbf{p}) \neq \emptyset$.

Debreu (1959: 44) states that "[g]iven a price vector **p**, it may be that there is no production vector that provides a maximum profit for a producer". The reason for this is that "if non-decreasing returns to scale prevail, and if for some $\mathbf{y}_k$ in Y_k one has $\mathbf{p} \bullet \mathbf{y}_k > 0$, profit can be arbitrarily increased".

The same difficulty is identified by Nikaido (1968: 252): "[o]ne typical example of factors that cause a no-profit situation is the cone property of the technology set." Consider the case where Y_k is a convex cone. In this case, we cannot guarantee $\pi_k(\mathbf{p}) \neq \emptyset$ because if $\mathbf{y}_k \in Y_k$, then $\boldsymbol{\Theta}\mathbf{y}_k \in Y_k$, for any $\boldsymbol{\Theta} > 0$. Then $\pi_k(\mathbf{p}) = \mathbf{p} \bullet \boldsymbol{\Theta}\mathbf{y}_k = \boldsymbol{\Theta}\, \mathbf{p} \bullet \mathbf{y}_k$ and this grows without limit as $\boldsymbol{\Theta} \rightarrow +\infty$. There is no production vector yielding the maximum profit.

If we want to be sure that $\pi_k(\mathbf{p}) \neq \emptyset$, we could use Weierstrass' theorem which states that if f is a continuous function from S to R, and if S is a compact non-empty set, then f(S) has a maximum and a minimum. But, as we have seen, there is in general no reason for the individual production possibility sets to be bounded – a property which is required for compactness.

A similar problem arises in the case of consumers: given a price vector **p** it may be that there are no most preferred consumption vectors for $\succsim$ among those that satisfy the budget constraint $\mathbf{p}\bullet\mathbf{x} \leq \mathbf{p}\bullet\mathbf{a} + \sum_{k=1} \alpha_{ik} \pi_k(\mathbf{p})$. In other terms, the properties of X_i are not enough to guarantee that $\phi_i(\mathbf{p}) \neq \emptyset$, i.e., the image set of the demand function is not the empty set.[4]

In the case of demand functions $\phi_i(\mathbf{p})$ over preference fields $(X_i, \succsim_i)$, the existence of a most preferred element is guaranteed if the subset M of consumption vectors satisfying the budget constraint is compact and if the preference ordering $\succsim$ is continuous.[5] But M cannot be compact unless X_i is compact. Thus, X_i cannot be an unbounded set.

It is important to note that the price mechanism cannot prevent firms from making unbounded commitments over their range of choices. Because agents are price takers, they must believe that for all semi-positive price vectors **p** they may buy and sell whatever amounts they choose to produce or consume. However, because the economy has an endowment of limited resources, a question of scarcity arises and unbounded commitments have to be ruled out. This disconnect between events that take place at the level of the individual agents and those that are intelligible at the aggregate level is described by Arrow and Hahn (1971: 63): “If Y_k is unbounded, then at a certain **p** it may be that the firm would like to produce on an infinitely large scale. This possibility, as such, does not make it impossible to conduct an analysis of market equilibrium with positive prices; although the firm is taken to suppose that it can sell and buy whatever quantities it likes at the going prices, the economy, in fact, may be incapable of producing outputs and using inputs in unlimited amounts. Indeed, if we are interested in a world of scarcity, we ought to exclude the possibility.”

But excluding this possibility is more difficult. It could be thought that the market mechanism takes care of this problem. Consider the following example of Marshallian inspiration: as a firm chooses to produce a very large quantity of certain outputs, it would exert a great pressure on the market for the required inputs. The prices of these inputs would rise, as the prices of the firm’s output would decrease, causing profit to vanish.

In Franklin Fisher’s terms (1983: 40) “there is a natural way” to deal with this problem of unbounded commitments: “That way uses what we know about the role of the price system in a world of limited resources. As resources become scarce, their prices ought to rise. Accordingly, it is possible to argue that unbounded commitments cannot be profitable since the unit costs of production would rise above the price at which output can be sold." Is this accurate? Unfortunately, in the context of our problem, it is not. The circularity of this claim is obvious: for prices to adjust in this manner, an aggregate excess demand function is needed, and this requires individual supply and demand functions to be defined. But this is precisely the problem that needs to be solved. If these functions are not defined, there is no aggregate excess demand and no price adjustment process: the market cannot perform the task we would expect it to carry out.

From the start, individual production and consumption possibility sets were endowed with the properties of convexity and closedness. But boundedness was not included among the original properties of possibility sets. Grave consequences follow (individual supply and demand functions are not defined), so why is boundedness not included among these original properties? Convexity and closedness are topological features that, in the context of the definition of the general equilibrium model, do not involve any reference to *quantitative* information. But boundedness is different as it requires a reference to quantitative magnitudes. The viability of production and consumption allocations for the entire economy is at stake if boundedness is treated carelessly.

We know that given a price vector **p**, certain economy-wide allocations may be impossible to attain. For example, a production plan may require inputs in amounts that will not or cannot be supplied by consumers. On the other hand, a consumption allocation may be unattainable because the goods that are required by it cannot be supplied by producers. So if individual production and consumption sets are to satisfy the condition of boundedness, this must be done in a manner such that the set of attainable allocations is not the empty set. In other words, it is not possible to simply assume that the individual consumption and production possibility sets are bounded sets and, at the same time, guarantee that the set of attainable allocations for all semipositive price vectors is different from the empty set. In other terms, the arbitrary introduction of boundedness for the individual possibility sets is a sufficient condition for the definition of individual supply and demand functions, but in and by itself it does not ensure that the set of attainable allocations is non-empty.[6]

Nikaido (1968: 257) proceeds to "substitute certain *virtual* supply and demand functions for the true ones" (emphasis in the original).[7] These virtual functions are to be defined by narrowing the ranges of consumers' and producers' choices to suitable bounded subsets of the original possibility sets. This is done in the following manner. We know that a competitive equilibrium, if it exists, must satisfy the following condition:

$$\mathbf{a} + \sum \mathbf{y}_k - \sum \mathbf{x}_i \in (\mathbf{a} + Y - X) \cap R_n^+$$

This enables us to define the following subsets of the original consumption and production possibility sets (we follow here Nikaido's approach; Arrow's and Debreu's manipulation of these original sets is the same):

$$\hat{X}_i = \{ \mathbf{x}_i \mid \mathbf{x}_i \in X_i, (\mathbf{a} + Y - \sum_{s \neq i} X_s - \mathbf{x}_i) \cap R_n^+ \neq \emptyset\} \text{ for } i = 1,...,l)$$

$$\hat{Y}_k = \{\mathbf{y}_k \mid \mathbf{y}_k \in Y_k, (\mathbf{a} + y_k + \sum_{t \neq k} Y_t - X_i) \cap R_n^+ \neq \emptyset\} \text{ for } k = 1,...,m)$$

These sets are non-empty, convex and bounded. However, they are not closed. Why? Because although closedness is a property conserved by aggregation, we do not know if the set of consumption menus and production vectors that satisfy the conditions in parenthesis are closed sets. To bring in this property, a sufficiently large cube E is chosen

$$E = \{ h \mid \xi_j \leq h \leq \eta_j (h = 1, ... , l)$$

such that $0, c_i \in E$, and $X^\wedge_i, Y^\wedge_k \subset E^o$ $(i = 1, ... , l, k = 1, ... , l)$.

New sets are then defined $Xi \cap E$, $Yk \cap E$ and these sets have all the desired properties which ensure that the individual supply and demand functions are defined for all semi-positive price vectors $\mathbf{p} \geq 0$: $\phi_i(\mathbf{p}) \neq \emptyset$., $\pi_k(\mathbf{p}) \neq \emptyset$. Aggregation of the individual supply and demand functions can now take place and an aggregate excess demand function can be defined.

The proof of existence of a competitive equilibrium now proceeds by applying Kakutani's fixed point theorem for upper semi-continuous correspondences to a suitable mapping. This mapping is formed by the Cartesian product of two mappings, the aggregate excess demand correspondence and a price adjustment mapping.[8] The aggregate excess demand function $\chi(p)$ is defined as follows:

$$\varphi(p) = a + \sum_k^m \varphi_k(p)$$
$$\phi(p) = \sum_i^l \phi_i(p)$$
$$\chi(p) = \varphi(p) - \phi(p)$$

Although it is now possible to aggregate the outcomes of individual agents' behavior, an important number of difficult questions pertaining to the nature of the individual agents and the scope of the existence proof must be addressed.

The general equilibrium model attempts to describe a private, decentralized economy, in which each agent engages in self-aggrandizing behavior. This is why the individual agents comprised by this model only possess private information about their preferences or their technology. The possibility sets X_i and Y_k correspond to this rationale. But the sets $X_i \cap E$ and $Y_k \cap E$ are not intelligible from this theoretical perspective. They represent an intersection between the private world of individuals and the realm of aggregate information that only a supra-individual agent or institution can possess.

In the proof of existence of a general competitive equilibrium the initial sets X_i and Y_k are substituted by $X_i \cap E$, $Y_k \cap E$ in order to ensure that the individual demand and supply functions are defined. But through this procedure, the specification of the individual agents changes: they are no longer specified by the purely private information contained in each set X_i and Y_k. The sets $X_i \cap E$, $Y_k \cap E$ involve information that is not known to the individual agents. Although it makes perfect mathematical sense to define these intersections, from the economic standpoint this involves a deep problem for the model. The agents are now required to have information about the boundaries of their individual sets, which have now been "appropriately" restricted. If individual agents were to know that the range of their individual choices is now restricted in this manner, this is equivalent to having them transcend the realm of their private worlds and use information that is the result of an aggregation process.

The sets $X_i \cap E$ and $Y_k \cap E$ represent impossible intersections between the private realm of the individual agents and the world of aggregate information. The former comprises information that is only available to each individual agent; in fact, it is not an abuse of language to say that the individual agents are made up of this information and the associated behavioral rules (i.e., the consumption and supply functions). But information about the aggregate possibility consumption and production sets, as well as information on the aggregate resource endowment of the economy can only be possessed by a supra-individual agent or authority.

Therefore, the intersections $X_i \cap E$ and $Y_k \cap E$ are not intelligible from the standpoint of each individual agent. In the context of the general equilibrium model, they represent an unfeasible juncture of the private (X_i, Y_k) and social (E) worlds. Through a manipulation over the original individual possibility sets that restricts the range of agents' choices, the individual demand and supply functions are defined, but the original agents are disfigured beyond recognition.

In other terms, the aggregate excess demand correspondence is unintelligible without the definition of the individual supply and demand functions. And these functions are defined only in the case of adequately constrained possibility sets $X_i \cap E$ and $Y_k \cap E$ which, in turn, cannot be known by the individual agents denoted through the indexes i and k. Thus, the aggregate excess demand function itself is a construct that clashes with the notion of a private decentralized economy.

At this point, the dilemma facing general equilibrium theory can be stated as follows: either we assume that individual possibility sets are (adequately) bounded and we have well defined individual supply and demand functions, but end up with a model that cannot be described as a representation of a private decentralized economy; or we consider unbounded individual possibility sets which conform to the notion of decentralized economy, but lose the possibility of having defined supply and demand functions. In this case, the obvious consequence is the impossibility of defining an aggregate excess demand function.

In the first horn of the dilemma, the construction of the model contradicts the object of the theory. But in the second horn we lose the possibility of having an aggregate excess demand function and using it for the proof of existence of equilibrium. This would lead to a dead end street. Given what's at stake here, it is not surprising that the authors of this theory chose the first alternative.

The proof of existence proceeds after the manipulation of the individual possibility sets to find a fixed point (interpreted as a general equilibrium) which possesses the desired property $u \in \chi(p)$ with $\chi(p) \cap R_n^+ \neq \emptyset$ (i.e., at the fixed point excess demand is negative). It is easy to verify that the individual choices of production vectors $\mathbf{y_k}$ and consumption menus $\mathbf{x_i}$ are elements of the original individual possibility sets Y_k, X_i. However, this is not enough to conclude that the proof is consistent with the definition of the individual agents. The individual choices y_k and x_i are the result of well specified behavioral rules which depend on the manipulation that is required by the model. But those rules are incompatible with the specification of individual agents in the context of a decentralized, private economy. To conclude, the proof of existence concerns a state of the economy that cannot be attained by the individual actions of the self-aggrandizing and decentralized agents originally specified for the general equilibrium model.

CONCLUDING REMARKS

The building blocks of general equilibrium theory, epitomized in the Arrow-Debreu model, have received very little attention. A thorough examination of the central issues surrounding the primitive concepts needed to put together the model has been missing. This is rather surprising given the fact that the Arrow-Debreu model is the workhorse of the theory of decentralized economies. Its building blocks deserve undivided attention because their coherence conditions the scope and legitimacy of the model's main results.

There is not much merit in debating the soundness of these primitive concepts or building blocks unless it is shown that these are relevant issues for the model's performance, i.e., its ability to deliver the expected results. This is of course the ultimate rationale for this discussion. And the model's performance is a function of both the mathematical accuracy and the consistency with the theoretical objectives. Using concepts of prices and commodities that clash with the definitions of goods and services (as physical entities, geographically and temporally determined) poses serious analytical questions that have been largely ignored. The questions surrounding boundedness of individual possibility sets leads to another formidable set of problems.

It is conceivable - indeed, it has been argued - that the issues raised by a more serious scrutiny of the concepts of prices and commodities might be merely a by-product of the necessary simplification that every model of reality entails. This view is wrong and should be abandoned. It is now time to come to grips with the true problems instead of sweeping them under a rug of rhetoric about the formidable mathematical tools that are used. What is the value of an existence proof that depends on commodities that cannot be measured, and “decentralized” agents whose behavior depends on a *deus ex machina* beyond their individual knowledge?

In attempting to rely on the purity of mathematical discourse, economic theory has frequently sacrificed content for the sake of using mathematical tools (Koopmans, 1957, was one of the first to point this out). The current state of applied economics is perhaps one consequence of this state of affairs. All models have to simplify the real world. But the issues examined in this chapter go beyond this problem. It is not a matter of oversimplification that has absorbed our attention. It is a matter of fundamental incompatibility between the building blocks and the house they help build.

ENDNOTES

[1] Much of the material presented in this chapter comes from two earlier publications. The first part is based on the results in Nadal (1984), while the second distills the main results found in Nadal and Salas (1987).
[2] A typical example here is the work of Uzawa (1962) showing how the convexity property for individual consumption and production possibility sets can be replaced by a less restrictive assumption of convexity of the aggregate possibility set for the entire economy.
[3] In the case of closedness, the aggregate set Y is not necessarily closed even if every Y_k is closed. However, if every Y_k is closed and convex, and if $Y \cap (-Y) = \{0\}$, then Y is closed (Debreu 1959:41). In addition, the assumption of convexity for every Y_k can be relaxed and substituted by convexity of the aggregate set Uzawa, H. (1962) 'Aggregative Convexity and the Existence of Competitive Equilibrium', *Economic Studies Quarterly*, **12:** 52-60..

[4] Both Debreu and Nikaido recognize that given a price vector $\mathbf{p} \geq 0$ there may be no consumption bundles satisfying the budget constraint. Even if that part of the problem is solved by introducing an assumption over the initial holdings of the individual agents and the budget constraint is consistent with the consumption set, given a price vector $\mathbf{p} \geq 0$ the existence of a most preferred consumption bundle is not guaranteed.
[5] See lemma 15.3 in Nikaido (1968).
[6] For a discussion on feasible allocations, see Nikaido (1968:247). Many microeconomic texts avoid the problem by assuming up front that the individual production and consumption possibility sets are bounded, without explaining how this property is introduced: Lancaster (1971), Malinvaud (1975), Quirk and Saposnik (1968), Varian (1980). Other authors like Weintraub (1982) and Takayama (1974) erroneously affirm that boundedness is guaranteed through lower boundedness and the budget constraint.
[7] In the chapters on individual producers and consumers, Debreu (1959:44, 62-63) follows a different sequence, restricting the subset of price vectors for which individual supply and demand functions are defined. The assumptions needed to guarantee this result are presented in the chapter on existence of a general equilibrium and they are equivalent to the procedure followed by Nikaido.
[8] In the next chapter we analyze the price adjustment mapping in detail.